

## Contact process on a Voronoi triangulation

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We study the continuous absorbing-state phase transition in the contact process on the Voronoi-Delaunay lattice. The Voronoi construction is a natural way to introduce quenched coordination disorder in lattice models. We simulate the disordered system using the quasistationary simulation method and determine its critical exponents and moment ratios. Our results suggest that the critical behavior of the disordered system is unchanged with respect to that on a regular lattice, i.e., that of directed percolation.

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### I. INTRODUCTION

Nonequilibrium phase transitions between an active (fluctuating) state and an inactive, absorbing state arise frequently in interacting particle models [1], chemical catalysis [2], interface growth [3], epidemics [4], and related fields. In spatially extended systems, exemplified by the contact process [5], such transitions are currently of great interest, which has been heightened by recent experimental confirmations of absorbing-state phase transitions in a liquid crystal system [6] and in a sheared colloidal suspension [7]. Much of this work is focused on issues of universality, aimed at characterizing the critical behavior of these models in terms of universality classes [1,8–10]. It has been conjectured [11,12] that models with a positive one-component order parameter, short-range interactions, and absence of additional symmetries or quenched disorder belong generically to the universality class of directed percolation (DP), which is considered the most robust universality class of transitions to an absorbing state.

The contact process (CP) is one of the simplest and most studied models belonging to the DP universality class. Of particular interest is how spatially quenched disorder affects its critical behavior [13]. Quenched disorder, in the form of impurities and defects, plays an important role in real systems and may be responsible for the rarity of experimental realizations of DP [14]. Quenched disorder in the contact process on a regular lattice has been studied in the forms of random deletion of sites or bonds [15–17] and of random spatial variation of the control parameter [18–20]. All these studies report a change in the critical behavior of the model. These findings are consistent with Harris' criterion [21], which states that quenched disorder is a relevant perturbation if

$$d\nu_{\perp} < 2, \quad (1)$$

where  $d$  is the dimensionality and  $\nu_{\perp}$  is the correlation length exponent of the pure model. [In DP this inequality is satisfied in all dimensions  $d < 4$ , since  $\nu_{\perp} = 1.096\ 854(4)$ ,  $0.734(4)$ , and  $0.581(5)$ , for  $d = 1, 2$ , and  $3$ , respectively [22–24].] Some

controversy remains whether the exponents change continuously with degree of disorder [16,25] or whether they change abruptly to the values in the strong disorder limit corresponding to the universality class of the random transverse Ising model, as suggested by Vojta in a recent work [19].

Harris' criterion determines the relevance of disorder in the form of independent random dilution (of sites and/or bonds) in a regular lattice. A somewhat different situation arises when the underlying graph is not periodic, as is the case in a deterministic aperiodic structure, or in a graph with a random neighbor structure such as the Voronoi triangulation. To determine the relevance of disorder in these cases, the following heuristic extension of Harris' criterion was proposed by Luck [26]: Consider a spherical patch  $\Omega$  with radius  $R$  on a given realization of a graph. The patch encloses a number  $B(R)$  of vertices, which scales as  $B(R) \sim R^d$ . The average coordination number in the patch is given by

$$J(R) = \frac{1}{B(R)} \sum_{i \in \Omega} q_i. \quad (2)$$

Let the fluctuation of the coordination number around its expected value,  $J_o = \bar{q}$ , decay as

$$\sigma_R(J) = \frac{\langle |J(R) - J_o| \rangle}{J_o} \sim \langle B(R) \rangle^{-(1-\omega)} \sim R^{-d(1-\omega)}, \quad (3)$$

when  $R \rightarrow \infty$ . Here,  $\omega$  is defined as the *wandering exponent* of the triangulation. Nearby the critical point  $\Delta \equiv (\lambda - \lambda_c)/\lambda_c = 0$ , the fluctuations  $\sigma_{\xi}(J)$  of the average coordination number in a correlation volume scale as

$$\sigma_{\xi}(J) \sim \xi_{\perp}^{-d/2} \sim \Delta^{\nu_{\perp}d/2}, \quad (4)$$

since  $\xi_{\perp} \sim \Delta^{-\nu_{\perp}}$ . Considering a large correlation volume,  $R \sim \xi_{\perp}$ , the resulting shift of the critical point, induced by the fluctuations  $\sigma_{\xi}$  in a correlation volume, is proportional to  $\Delta^{d\nu_{\perp}(1-\omega)}/\sqrt{\text{var}(q_i)}$ , where  $\text{var}(q_i) = \langle q_i^2 \rangle - \langle q_i \rangle^2$ . Then, in order that the regular critical behavior remain unchanged, these fluctuations should die out when  $\Delta \rightarrow 0$ , which is true if  $\omega$  does not exceed a threshold value given by

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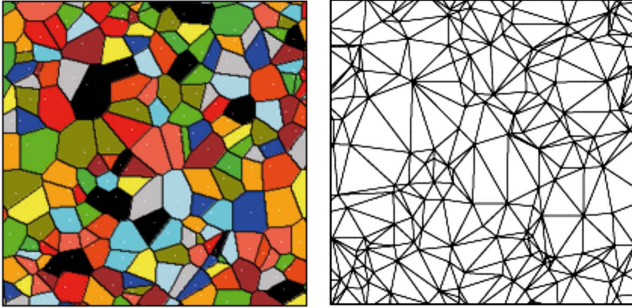


FIG. 1. (Color online) (a) A patch of a Voronoi diagram. (b) The corresponding dual lattice to the diagram shown in (a).

$$\omega_c = 1 - \frac{1}{d\nu_{\perp}}. \quad (5)$$

Thus, in principle, the Harris-Luck criterion permits one to predict the effects of quenched disorder in models defined on structures such as quasicrystals or even random lattices. (Note that for independent dilution,  $\omega=1/2$ , and Luck's expression reduces to the Harris criterion.)

In this work we investigate whether disorder in the form of a quenched Poissonian *coordination disorder* alters the critical behavior of the contact process by studying the critical behavior of the process on a Voronoi-Delaunay (VD) type random lattice [27,28]. The VD lattice represents a natural way of introducing quenched coordination disorder in a lattice model and also plays an important role in the description of idealized statistical geometries such as planar cellular structures, soap throats, etc. [27]. In this lattice, the sites are spatially distributed following a Poisson distribution, and the coordination number  $q$  varies randomly, with  $3 \leq q < \infty$  and  $\bar{q}=6$  in the infinite-size limit. Our results suggest that coordination disorder does not change the critical behavior of the contact process.

The balance of this paper is organized as follows. In the next section we review the definition of the contact process and detail construction of the VD lattices as well the simulation methods used. In Sec. III we present our results and discussion; Sec. IV is devoted to our conclusions.

## II. MODEL AND METHOD

Consider a bounded domain  $\Omega$  in a  $d$ -dimensional space in which  $N$  nodes are randomly placed with uniform distribution. The Voronoi diagram of this set is a subdivision of the domain into regions  $V_i$  (with  $i=1, 2, \dots, N$ ), such that any point in  $V_i$  is closer to node  $i$  than to any other node in the set. Figure 1(a) shows a patch of a Voronoi diagram. The points whose cells share an edge are considered neighbors. The dual lattice, obtained by linking neighboring sites is the Voronoi-Delaunay network, exemplified in Fig. 1(b). One of the characteristics of the dual lattice is that its local coordination number varies randomly, with the distribution shown in Fig. 2. In this work, we take periodic boundary conditions, i.e., the domain  $\Omega$  has a toroidal topology. In order to construct the lattices we follow the method of Ref. [29]. For

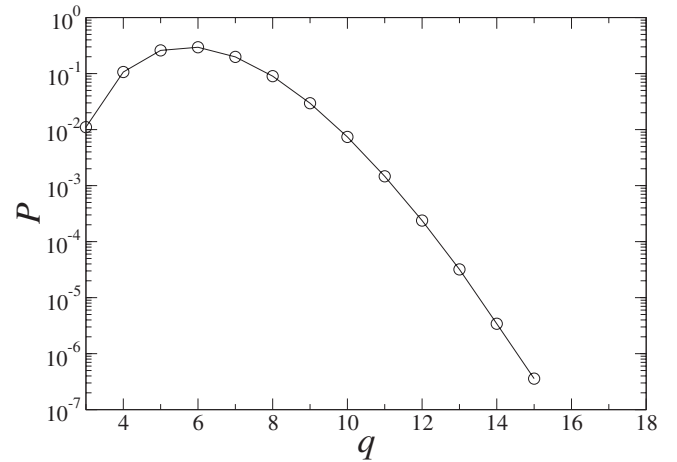


FIG. 2. Degree distribution  $P(q)$  of the Voronoi-Delaunay lattice, for system size  $L=2560$ .

simplicity, we express the length  $L$  of the domain  $\Omega$  in terms of the size of a regular lattice  $L=\sqrt{N}$ .

The CP, originally introduced as a “toy model” for epidemic spreading [5], is a stochastic interacting particle system defined on a lattice, with each site either occupied [ $\sigma_i(t)=1$ ], or vacant [ $\sigma_i(t)=0$ ]. Transitions from  $\sigma_i=1$  to  $\sigma_i=0$  occur at a rate of unity, independent of the neighboring sites. The reverse transition is only possible if at least one of its neighbors is occupied: the transition from  $\sigma_i=0$  to  $\sigma_i=1$  occurs at rate  $\lambda r$ , where  $r$  is the fraction of nearest neighbors of site  $i$  that are occupied; thus the state  $\sigma_i=0$  for all  $i$  is absorbing. ( $\lambda$  is a control parameter governing the rate of spread of activity.)

In the simulation we employ the usual simulation scheme [1], in which annihilation events are chosen with probability  $1/(1+\lambda)$  and creation with probability  $\lambda/(1+\lambda)$ . In order to improve efficiency, the sites are chosen from a list of currently occupied sites. In the case of annihilation, the chosen site is vacated, while, for creation events, one of its  $q$  nearest-neighbor sites is selected at random and, if it is currently vacant, it becomes occupied. The time increment associated with each such event is  $\Delta t=1/N_{occ}$ , where  $N_{occ}$  is the number of occupied sites just prior to the attempted transition.

In the studies reported here we sample the *quasistationary* (QS) distribution of the process (that is, conditioned on survival), which has proven a very useful tool in the study of processes with an absorbing state [1,30,31]. For this purpose, we employ a simulation method that yields quasistationary (QS) properties directly, the QS simulation method [32]. The method is based in maintaining, and gradually updating, a set of configurations visited during the evolution; when a transition to the absorbing state is imminent the system is instead placed in one of the saved configurations. Otherwise the evolution is exactly that of a conventional simulation.

## III. RESULTS AND DISCUSSION

We performed extensive simulations of the CP on Voronoi-Delaunay random lattices of  $L=20, 40, \dots, 640$ , us-

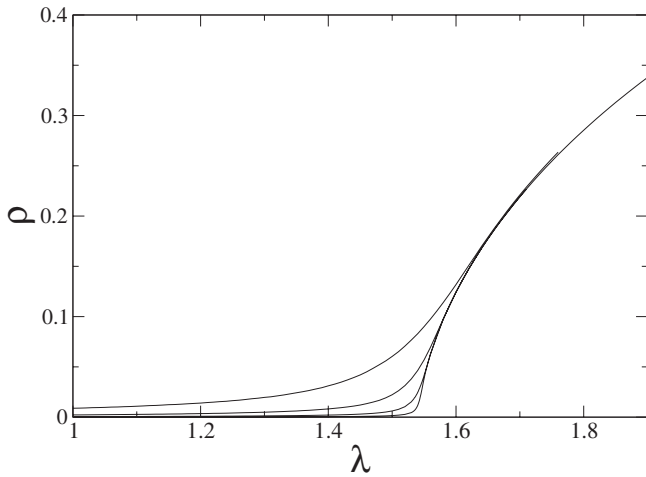


FIG. 3. Quasistationary density of active sites  $\rho$  as a function of the control parameter  $\lambda$ . System sizes:  $L=20, 40, 80$ , and  $160$ , from top to bottom.

ing the QS simulation method. Each realization of the process is initialized with all sites occupied, and runs for at least  $10^8$  time steps. Averages are taken in the QS regime, after discarding an initial transient which depends on the system size. This procedure is repeated for each realization of disorder (for each size studied, we performed averages over 200–300 different lattices).

In Fig. 3 we show the quasistationary density  $\rho$  as a function of the control parameter  $\lambda$  for several values of  $L$ . We observe, as expected, a continuous phase transition from an active to an absorbing state. Since, due to topological constraints, the Voronoi-Delaunay lattice possesses  $\bar{q} \approx 6$ , the value of the critical point is shifted from  $\lambda_c = 1.648\,77(3)$  [33] (regular square lattice) to  $\lambda_c = 1.542\,66(4)$  (the increase in  $q$  facilitates creation). This is very close to the critical value,  $\lambda_c = 1.547\,80(5)$ , for the regular triangle lattice, obtained using the same methods as described below. It is notable that the critical value of the disordered system is about 0.3% smaller than that of the regular lattice with the same

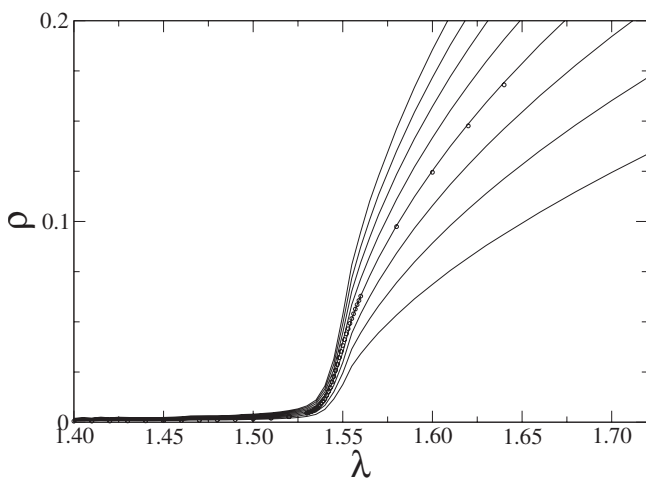


FIG. 4. QS density of active sites  $\rho(q)$  vs  $\lambda$ , for sites with  $q=3, 4, \dots, 10$ , from bottom to top. Circles: average over all sites. System size  $L=160$ .

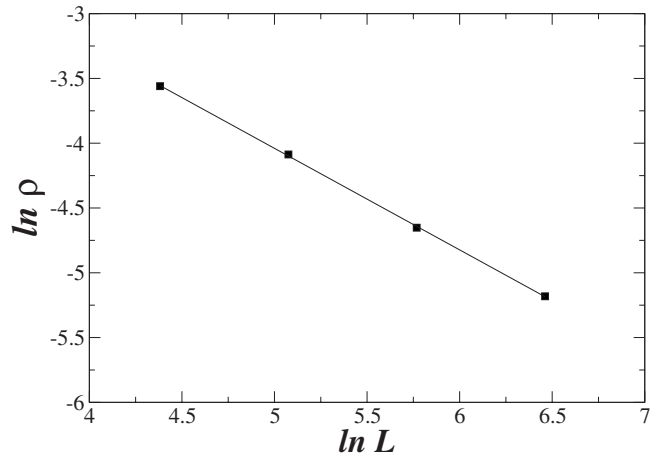


FIG. 5. QS order parameter  $\rho$  vs system size  $L$  at criticality ( $\lambda=1.542\,66$ ).

average connectivity. Figure 4 shows how the QS density of active sites varies with the coordination number  $q$ .

At the critical point we find that the quasistationary density decays as a power law,  $\rho \sim L^{-\beta/\nu_\perp}$ , as shown in Fig. 5. Our simulation data follow a power law with the exponent  $\beta/\nu_\perp = 0.791(7)$ , while the value for DP in two spatial dimensions is  $0.797(3)$  [33].

Another important quantity is the lifetime of the QS state,  $\tau$ . In QS simulations we take  $\tau$  to be the mean time between successive attempts to visit the absorbing state. Figure 6 shows that at the critical point, the lifetime also follows a power law,  $\tau \propto L^z$ , with  $z = \nu_\parallel/\nu_\perp = 1.78(3)$ , as compared with the literature value of  $1.7674(6)$  for the DP class [33].

Complete characterization of a nonequilibrium universality class requires the determination of at least three independent critical exponents. To this end we perform initial decay studies on large systems, starting with a fully occupied lattice. While the CP with random dilution exhibits a logarithmic relaxation [16], on the VD lattice we observe a clear power-law decay. Finite size scaling in this case predicts that  $\rho \sim t^{-\delta}$ . A least-squares fit for the data shown in Fig. 7 yields  $\delta = 0.453(9)$ , in very good agreement with the standard value of  $\delta = 0.4523(10)$  for DP [33].

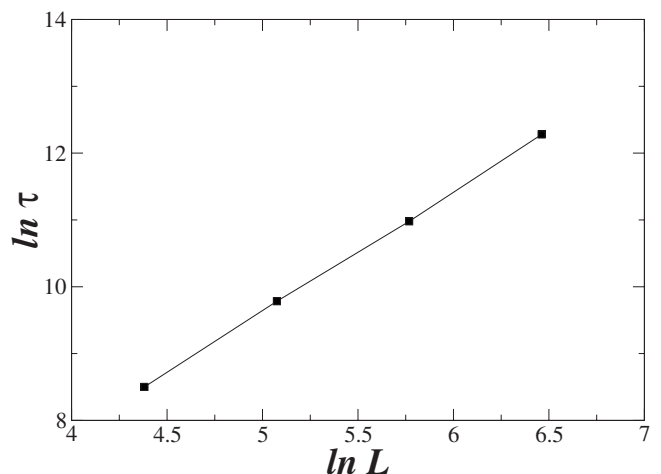


FIG. 6. Critical lifetime of the QS state  $\tau$  vs  $L$ .

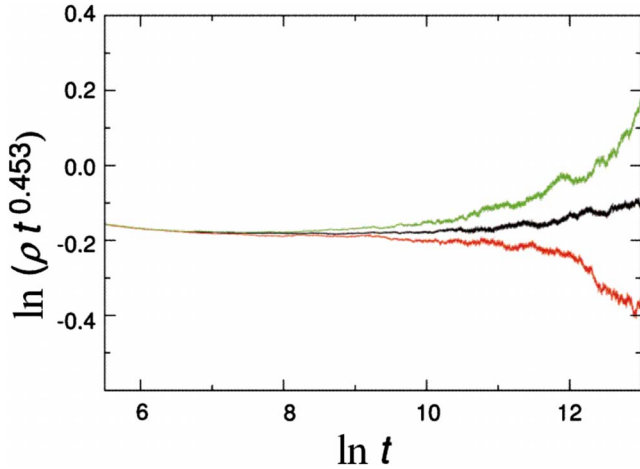


FIG. 7. (Color online) Decay of the order parameter starting from a full lattice of size  $L=2560$ .  $\lambda=1.5428$ ,  $1.5427$ , and  $1.5426$ , from top to bottom.

Moment ratios (or reduced cumulants) represent an alternative method for identifying the universality class of a continuous phase transition [34–36]. Here we analyze the critical moment ratio  $m = \langle \rho^2 \rangle / \langle \rho \rangle^2$ . This quantity is analogous to Binder’s reduced fourth cumulant [37] at an equilibrium critical point: the curves  $m(\lambda, L)$  for various  $L$  cross near  $\lambda_c$  (the crossings approach  $\lambda_c$ ), so that  $m$  assumes a universal value  $m_c$  at the critical point, as can be seen in Fig. 8. In this case, our data yield a universal value of  $m_c = 1.328(6)$ , again in very good agreement with the best known value for the CP on a regular square lattice,  $m_c = 1.3257(5)$  [34].

In summary, our results reveal that the absorbing phase transition of the contact process defined on the Voronoi-Delaunay random lattice belongs to the directed percolation universality class. These results are somewhat surprising, since the wandering exponent for these lattices was numerically evaluated in an extensive work by Janke and Weigel [38], who found that  $\omega = 1/2$ , i.e., the relevance criterion for such lattices reduces to the usual Harris criterion, Eq. (1).

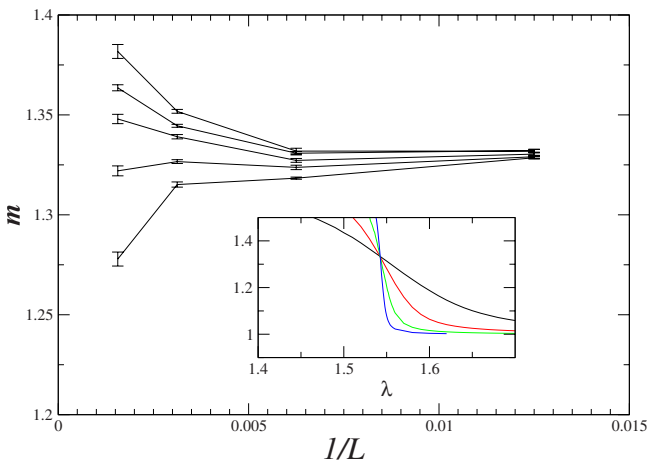


FIG. 8. (Color online) Quasistationary moment ratio  $m$  vs  $\ln L$ , for  $\lambda = 1.54256$ ,  $1.54260$ ,  $1.54264$ ,  $1.54268$ , and  $1.54280$ , from top to bottom. System size:  $L=640$ . Inset: Quasistationary moment ratio  $m$  vs  $\rho$ , system sizes:  $L=20, 40, 80$ , and  $160$ .

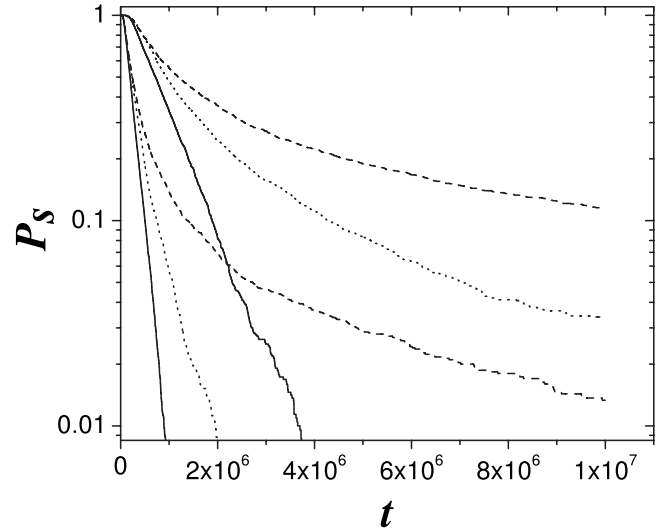


FIG. 9. Survival probability vs time, in the critical CP on a Voronoi lattice (solid line), and for the critical CP on a square lattice with random dilution of 2% (dotted) and 5% (dashed). System sizes:  $L=640$  (left curves) and  $L=1280$  (right).

In the equilibrium context the Harris-Luck criterion has been verified numerically on random lattices in several models, such as the Ising model [39,40] and percolation [41]. However, Monte Carlo simulations for the  $q=3$  Potts model [42] as well for the Ising model in 3D [43,44] and for the spin-3/2 Blume-Capel model [45] yield results that contradict the relevance threshold given by the Harris-Luck criterion. Simulation results for nonequilibrium models, viz. the majority-vote model on a random lattice [46], also appear to contradict this relevance criterion.

In Refs. [38,44] it is suggested that Voronoi disorder appears not to alter the critical behavior because it is intrinsically weak, and that the usual hallmarks of quenched disorder would in fact manifest themselves in larger systems. In order to test this hypothesis, we compare in Fig. 9 the survival probability  $P_s$  (starting with a fully occupied lattice) of the CP on the VD lattice and on a regular (square) lattice with weak dilution. It is known that the diluted CP exhibits activated disorder [19], due to emergence of favorable regions, leading to logarithmically slow dynamics [16]. We find that while in the CP on the VD lattice the survival probability decays exponentially (as in the ordinary contact process), in the diluted CP the behavior is clearly different.

Notice that the effect of the “rare regions” is clearly visible for the system sizes used here, even for the smallest dilution (2%): the decay of the survival probability is clearly slower than exponential. On the square lattice with dilution  $x \ll 1$ , the variance of the connectivity  $\text{var}(q) \approx 4x$ , so that  $\text{var}(q) \approx 0.08$  for dilution 0.02. This is less than 5% of that for the VD lattice, where  $\text{var}(q) = 1.779(2)$ . We also should mention that the effects of quenched disorder in CP are stronger than in the three-dimensional Ising model: in the latter, quenched disorder provokes a difference in the second digit in the exponent  $\nu$  [47], while for models in the DP class even weak disorder changes the critical dynamic properties drastically [48].



## IV. CONCLUSIONS

We performed large-scale simulations of the contact process on a Voronoi-Delaunay random lattice, which exhibits quenched connectivity disorder in the model. Our results suggest that this kind of disorder does not alter the DP character of the transition, in contradiction with the Harris-Luck criterion. Given the large systems sizes and long simulation times used, it appears unlikely that the system will eventually cross over to non-DP scaling. Thus it remains an open

question why an argument of the Harris-Luck type is not applicable in some cases. Our results also reveal that the DP universality class may be even more robust than asserted in the usual DP conjecture, in the sense that not all kinds of quenched disorder are relevant perturbations.

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